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Regularity of Linear Systems of Plane Curves

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INTRODUCTION

The subject of this paper is the study of linear systems of plane curves which are given by assigning a set of fixed multiple points. This study not only has an interest *per se*, but is connected to other subjects (for example, results on rational surfaces and curves on them can be found in [Ro], [A], and [Gi]).

A linear system of plane curves of (projective) dimension r is a subvector space $I_t \subseteq R_t$ of the vector space of forms of degree t, where $R = \bigoplus_t R_t = k[x_0, x_1, x_2]$ is the ring of homogeneous coordinates in $\mathbb{P}^2_k$ and k is an algebraically closed field.

We shall consider systems I_t given by sets of base points, i.e., systems formed by all curves of degree t having multiplicities at least m_i at assigned points P_i with $i = 1, ..., s$ (so that each P_i will be at least an m_i-fold point for the system).

If we consider such systems for every t, we have that their direct sum, $I = \bigoplus_t I_t$, is a homogeneous ideal in R, of the form

$$I = \bigcap_{i=1}^{s} \mathbf{p}_i^{m_i} \subseteq k[x_0, x_1, x_2],$$

where $\mathbf{p}_i$ is the prime homogeneous ideal associated to the point P_i in the graded ring R. I defines a 0-dimensional subscheme Z of $\mathbb{P}^2$ which has support on the P_is and multiplicity $\binom{m_i+1}{2}$ at P_i.

An m-fold point imposes $\binom{m+1}{2} = m(m+1)/2$ conditions on curves, so, if all these conditions are independent, the (affine) dimension of our system should be

$$\dim R_t - \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2} = \binom{t+2}{2} - \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2}.$$

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This number is called the *virtual dimension* of the system I_t and when it is exactly the actual dimension of the vector space I_t the system is called *regular*.

The main problem we will study in this paper is: Given a set of distinct points $P_1, ..., P_s$ and multiplicities $m_1, ..., m_s$, find a bound for the integer

$$\tau = \tau(Z) = \min\{t \mid I_t \text{ is regular}\}.$$

One bound is classically known:

$$\tau \leqslant \sum_{i=1}^{s} m_i - 1,$$

where equality is achieved if and only if the points are on a line (see e.g. [DG1]).

With more hypotheses on the points, B. Segre in [S.2] gave two better bounds for τ, while B. Harbourne (see [H.1], [H, 2]) has given an algorithm to find out if a system is regular in the case of points contained in a curve of small degree d ($d \leqslant 3$; for $d = 1, 2$ see also [DG1] and [G]).

Our aim is to generalize some of the results by Harbourne to groups of generic points.

The bound we find is not sharp, but it is an improvement of the ones mentioned above. Namely, if $S = \{P_1, ..., P_s\} \subseteq \mathbb{P}^2$ is a set of distinct point and Z, $m_1, ..., m_s$ are as before, our result is:

THEOREM 1. *Let S be contained in the non-singular locus of an irreducible reduced curve $C \subseteq \mathbb{P}^2$ of degree $d \geqslant 3$ and suppose that $\tau(S) \geqslant d$. Then, if $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s \geqslant 0$,*

$$\tau(Z) \leqslant \sum_{i=1}^{d} m_i$$

unless $s = d(d+3)/2$ and $m_1 = m_2 = \cdots = m_s \geqslant 2$.

In the latter case, we have $\tau(Z) \leqslant \sum_{i=1}^{d} m_i + 1$.

The case when $\tau(Z) = \sum_{i=1}^{d} m_i + 1$ is analyzed in the last section and is related to the possibility that the divisor $\sum_{i=1}^{s} 2P_i$ ($\sum_{i=1}^{d} m_i P_i$ if $d = 3$) is a complete intersection on C.

In particular, if $A_d = \sum_{i=1}^{d} m_i$, we have $\tau(Z) \leqslant A_{\tau(S)+1}$.

Moreover, for generic $P_1, ..., P_s$ we have:

PROPOSITION 2. *For a generic set of points $P_1, ..., P_s \subseteq \mathbb{P}^2$, with $s \leqslant d(d+3)/2$, we have*

$$\tau(Z) \leqslant \sum_{i=1}^{d} m_i.$$

Theorem 1 is proved in Section 1 (rephrased in term of linear systems on the blow-up of $\mathbb{P}^2$ at the $P_1, ..., P_s$); the other results are given in Section 2.

This paper constitutes part of my Ph.D. thesis at Queen's University, and I thank my supervisor, Professor A. V. Geramita, for his help. Special thanks also are due to the referee whose detailed remarks and contributions I very much appreciated.

0. *Preliminaries*

The approach we shall use to study linear systems consists of "moving" from $\mathbb{P}^2$ to the surface X that we obtain by considering the blowing-up (X, π) of $\mathbb{P}^2$ at the points P_i.

In $\mathbb{P}^2$ we were dealing with a linear system I_t that was not *complete* (i.e., it did not contain every divisor linearly equivalent to the ones in it), but I_t corresponds to a complete linear system on X. Namely, if E_i are the exceptional divisors in X over P_i and E_0 is the strict transform of L, we can now study the complete linear system $|tE_0 - m_1 E_1 - \cdots - m_s E_s| = |D_t|$ on X, which has the same dimension as I_t (see e.g. [D] or [Ha]).

If $Z \subseteq \mathbb{P}^2$ is the scheme defined by $I = \bigoplus_t I_t$, we have that the condition "I_t regular" can be expressed in terms of the cohomology of the ideal sheaf of Z. Consider in fact $\mathcal{I}_Z$, $\mathcal{O}_Z$, the ideal and structure sheaves of Z; we have that $I_t \cong H^0(\mathcal{I}_Z(t))$.

From the exact sequence

$$(*) \qquad 0 \to \mathcal{I}_Z(t) \to \mathcal{O}_{\mathbb{P}^2}(t) \to \mathcal{O}_Z(t) \to 0,$$

(we use $h^i(\mathcal{F})$ to indicate $\dim_k H^i(\mathbb{P}^r, \mathcal{F})$, for every sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^r}$-modules) we obtain

$$h^0(\mathcal{I}_Z(t)) = h^0(\mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathcal{O}_Z(t)) + h^1(\mathcal{I}_Z(t)),$$

and since

$$h^0(\mathcal{O}_{\mathbb{P}^2}(t)) = \binom{t+2}{2} \qquad \text{and} \qquad h^0(\mathcal{O}_Z(t)) = \deg Z = \sum_{i=1}^{s} \binom{m_i+1}{2},$$

we have that $h^1(\mathcal{I}_Z(t)) = \dim I_t - \binom{t+2}{2} + \sum_{i=1}^{s} \binom{m_i+1}{2}$, hence $h^i(\mathcal{I}_Z(t)) = 0$ if and only if I_t is regular.

To each scheme Z of the type considered above there corresponds a divisor on the blow-up X, namely

$$E = m_1 E_1 + \cdots + m_s E_s,$$

as follows: since (see [Ha, II, Section 7]) $\mathcal{I}_Z = \mathcal{I}_1^{m_1} \cdots \mathcal{I}_s^{m_s}$, where $\mathcal{I}_i$ is the ideal sheaf associated to $\mathbf{p}_i$, and

$$\pi^{-1} \mathcal{I}_i^{m_i} = (\mathcal{I}_{E_i})^{m_i} = \mathcal{O}_X(-m_i E_i),$$

we have that

$$\pi^{-1}\mathcal{I}_Z = \mathcal{I}_E = \mathcal{O}_X(-m_1E_1 - \cdots - m_sE_s),$$

where we use $\mathcal{O}_X(D)$ to indicate the invertible sheaf on X associated to a divisor $D \in \operatorname{Div} X$.

We also have

$$h^i(\mathcal{I}_Z(t)) = h^i(\mathcal{O}_X(tE_0 - E)) \qquad i \geqslant 0.$$

The equality is well known for $i=0$ (see [D] or [Ha], already quoted), it is trivial for $i=2$ since both terms $=0$ (from 0.2 below and $(*)$), and for $i=1$ it can be computed from $(*)$ and the Riemann–Roch Theorem on X.

Recall the intersection theory on X: In $\operatorname{Pic} X = \bigoplus^{s+1}\mathbb{Z}$, the classes of the divisors $E_0, E_1, ..., E_s$ are a basis (as a $\mathbb{Z}$-module) and the intersection formulas are

$$E_0^2 = 1, \qquad E_i^2 = -1, \qquad E_i \cdot E_j = 0 \qquad \forall i \neq j.$$

We want to point out a few more facts that we will use in the sequel (see e.g. [H.1]):

(0.1) $\forall D \in \operatorname{Div} X$, $D \cdot E_0 < 0 \Rightarrow h^0(D) = 0$.

(0.2) $\forall D \in \operatorname{Div} X$, $D \cdot E_0 \geqslant -2 \Rightarrow h^2(D) = 0$.

(0.3) Let $D' \in \operatorname{Div} X$ be such that $\exists H \in |D'|$, H an irreducible and reduced curve. Then (Riemann–Roch on H) $\forall D \in \operatorname{Div} X$: $D \cdot D' \geqslant 2p_\alpha(H) - 1 \Rightarrow h^i(\mathcal{O}_H(D)) = 0$, while $D \cdot D' = 2p_\alpha(H) - 2$ and $h^i(\mathcal{O}_H(D)) \neq 0 \Rightarrow \mathcal{O}_H(D) = \omega_H$.

(0.4) For every invertible sheaf $\mathcal{F}$ on $\mathbb{P}^2$,

$$h^i(\mathbb{P}^2, \mathcal{F}) = h^i(X, \pi^*\mathcal{F}) \qquad \forall i \geqslant 0.$$

Throughout this work, the main method we use to determine the cohomology of a given divisor D is to split D into the sum of two divisors, $D = D' + D''$, of which one has in its class a reduced, irreducible curve, and the other has known cohomology (see also "Horace's Method" in [Hi]).

1. *d-Standard Divisors on Blow-ups of* $\mathbb{P}^2$

Let $P_1, ..., P_s$, $s > 0$, be distinct points of $\mathbb{P}^2$, and X, $E_0, E_1, ..., E_s$ be as in the introduction. Let $D \in \operatorname{Pic} X$, with $D = tE_0 - m_1E_1 - \cdots - m_sE_s$, and $d \geqslant 3$ be an integer.

In this section we introduce the concept of *d-standard* divisors on X, and study their properties.

DEFINITION. The divisor $D \in \operatorname{Pic} X$ is said to be a d-standard divisor if we have

$$t \geqslant \sum_{i=1}^{d} m_i \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s \geqslant 0.$$

Our main results will be:

THEOREM 1.1. *Suppose that $h^1(dE_0 - E_1 - \cdots - E_s) = 0$ and there exists a reduced and irreducible curve $C \in |dE_0 - E_1 - \cdots - E_s|$. Then if D is d-standard we have $h^1(D) = 0$, unless $D = m(dE_0 - E_1 - \cdots - E_s)$, with $m \geqslant 2$ and $s = d(d+3)/2$.*

PROPOSITION 1.2. *With the hypothesis and notation of Theorem 1.1 assume $h^1(D) \neq 0$, and let $n = \min\{m \mid h^1(mC) \neq 0\}$. Then $n = 2$ if $d > 3$ and, in any case, nC cuts out a canonical divisor on C.*

Remark. In our applications C will be the strict transform of an irreducible curve with a simple point at each P_i.

The proof of the Theorem relies on a series of lemmata:

LEMMA 1.3. *D is d-standard if and only if it is a combination, with non-negative integer coefficients, of divisors S_i for $i = 0, 1, \ldots, s$, where $S_0 = E_0$; $S_1 = E_0 - E_1$; $S_2 = 2E_0 - E_1 - E_2$; $S_3 = 3E_0 - E_1 - E_2 - E_3; \ldots$; $S_{d-1} = (d-1)E_0 - E_1 - \cdots - E_{d-1}$; and, for $d \leqslant i \leqslant s$, $S_i = dE_0 - E_1 - \cdots - E_i$.*

Proof. Obviously each of the S_i, $i = 0, \ldots, s$ is d-standard, so every combination of them with non-negative coefficients is d-standard also.

Now let D be d-standard; if $m_1 = \cdots = m_s = 0$ then either D is the 0-divisor or $D = tE_0$, so we are done in both cases.

Otherwise $m_i > 0$ for some i; let j be the largest integer such that $m_j > 0$. If $1 \leqslant j \leqslant d-1$, we can write

$$D = \left(t - \sum_{i=1}^{j} m_i E_i\right) E_0 + (m_1 - m_2)(E_0 - E_1)$$
$$+ (m_2 - m_3) S_2 + \cdots + (m_{j-1} - m_j) S_{j-1} + m_j S_j$$

and we are done again.

Hence the only problem is now for $j \geqslant d$. We proceed by induction on j. If $j \leqslant d-1$ we are already done, so let $j \geqslant d$ and consider

$$D - m_j S_j \cong (t - dm_j) E_0 - (m_1 - m_j) E_1 - \cdots - (m_{j-1} - m_j) E_{j-1}.$$

This is again a d-standard divisor, and by the induction hypothesis (since its jth coefficient $=0$) it is a combination, with non-negative coefficients, of the S_is.

But $D=(D-m_jS_j)+m_jS_j$, so we are done. ∎

Remark 1.4. If D is d-standard, then $h^2(D)=0$ by (0.2). Moreover, if we suppose $h^0(S_s)>0$, then $h^0(D)>0$ too, i.e. there is an effective divisor in $|D|$. In fact $h^0(D)>0$ follows from Lemma 1.3, since there is an effective divisor in any linear system $|S_i|$, $i=0, ..., s$.

It is easy to check this last assertion for $1\leqslant i\leqslant d-1$ considering curves in $\mathbb{P}^2$ through the points $P_1, ..., P_{d-1}$, while for $d\leqslant i\leqslant s$ it comes from our initial hypothesis, $h^0(S_s)>0$.

LEMMA 1.5. *Let D be d-standard, and let $j=\max\{i\,|\,m_i>0\}$. If there is a reduced irreducible curve $C\in|dE_0-E_1-\cdots-E_s|$, we have that if $j\leqslant 3d-1$, then $h^1(D)=0$.*

Proof. If D is the 0-divisor we have (by (0.4))

$$h^1(D)=h^1(\mathcal{O}_X)=h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})=0,$$

and we are done.

Otherwise, by Lemma 1.3, D can be written

$$D\cong b_0E_0+b_1S_1+b_2S_2+\cdots+b_jS_j,$$

where $b_i\geqslant 0$, $i=0, ..., j-1$ and $b_j>0$.

We proceed by induction on the number $t=\sum_{i=1}^{j} b_i$; if $t=0$, then $D=0$ and we have seen that $h^1(D)=0$.

If $t>0$, we can suppose $j=s$ (by (0.4), the cohomology of D can be computed on the blow-up of $\mathbb{P}^2$ at $P_1, ..., P_j$). By the induction hypothesis ($D-S_j$ is d-standard since $b_j>0$) we then have $h^1(D-S_j)=0$.

Now consider the exact sequence

$$0\to\mathcal{O}_X(-S_j)\to\mathcal{O}_X\to\mathcal{O}_C\to 0.$$

It yields

$$0\to\mathcal{O}_X(D-S_j)\to\mathcal{O}_X(D)\to\mathcal{O}_C(D)\to 0$$

and, passing to cohomology,

$$0=H^1(D-S_j)\to H^1(D)\to H^1(\mathcal{O}_C(D))\to H^2(D-S_j)=0.$$

So if we can prove that $h^1(\mathcal{O}_C(D))=0$ we shall be done.

We want to use (0.3) (for $D'=S_j$), and to be able to do so we have to show that $S_j\cdot D\geqslant 2p_a(C)-1$.

We have $p_\alpha = p_\alpha(C) = \frac{1}{2}(S_j^2 + K_X \cdot S_j) + 1$, so

(a) If $j = 0$, i.e., if $S_j = E_0$, then $p_\alpha = 0$, hence

$$D \cdot S_j = D \cdot E_0 = t \geqslant -1 = 2p_\alpha - 1.$$

(b) If $1 \leqslant j \leqslant d-1$, then $2p_\alpha - 1 = j^2 - j - 3j + j + 1 = j^2 - 3j + 1$ and we want

$$D \cdot S_j = jt - \sum_{i=1}^{j} m_i \geqslant 2p_\alpha - 1 = j^2 - 3j + 1,$$

i.e.,

$$jt - j^2 - \sum_{i=1}^{j} m_i + 3j - 1 \geqslant 0.$$

Since $t \geqslant \sum_{i=1}^{j} m_i$, this will be true if

$$jt - j^2 - t + 3j - 1 \geqslant 0,$$

i.e., if $t(j-1) - j^2 + 3j - 1 \geqslant 0$, which is trivially true, since $t \geqslant j > 0$.

(c) If $d \leqslant j \leqslant 3d-1$, then $2p_\alpha = d^2 - j - 3d + j + 1$, and we want

$$D \cdot S_j = dt - \sum_{i=1}^{j} m_i \geqslant d^2 - 3d + 1.$$

Let $t = d + k$. Then $dt - \sum_{i=1}^{j} m_i$ is minimum for $j = 3d-1$ and in this case $\sum_{i=1}^{j} m_i = (3d-1)(t/d) = ((3d-1)(d+k))/d$. So we shall be done if

$$d(d+k) - \frac{(3d-1)(d+k)}{d} - d^2 + 3d - 1 \geqslant 0;$$

i.e.,

$$d^3 + d^2k - 3d^2 + d - 3dk + k - d^3 + 3d - d \geqslant 0;$$

i.e.,

$$dk(d-3) + k \geqslant 0,$$

which is always true, since $d \geqslant 3$. ∎

LEMMA 1.6. *Let $h^1(dE_0 - E_1 - \cdots - E_s) = 0$, and*

$$D = tE_0 - m_1 E_1 - m_2 E_2 - \cdots - m_k E_k - E_{k+1} - \cdots - E_s,$$

where $m_i \geqslant 1$ $\forall i$ *and* $k<d$. *If* D *is d-standard* (*so* $t \geqslant m_1 + \cdots + m_k + d - k$), *we have*

$$h^1(D)=0.$$

Proof. We work by induction on k. We first consider the case when $k=1$, i.e., when $D=tE_0-mE_1-E_2-\cdots-E_s$, with $t \geqslant m+d-1$, and we can work by induction on m.

If $m=1$ we are done by hypothesis. So let $m \geqslant 2$ and consider $D=D'+D''$, where $D''=(t-1)E_0-(m-1)E_1-E_2-\cdots-E_s$ and $D'=E_0-E_1$.

We have that D'' has $t-1 \geqslant (m-1)+d-1$, because $t \geqslant m+d-1$, so by the induction hypothesis $h^i(D'')=0$.

Let $L \in |E_0-E_1|$ be the strict transform of a line through P_1; we have the exact sequence

$$0 \to \mathcal{O}_X(-D') \to \mathcal{O}_X \to \mathcal{O}_L \to 0.$$

Tensoring with D, we get

$$(*) \qquad 0 \to \mathcal{O}_X(D'') \to \mathcal{O}_X(D) \to \mathcal{O}_L(D) \to 0.$$

Since L is rational and $\deg \mathcal{O}_L(D)=L \cdot D=t-m>0$ (because $t \geqslant m+d-1>m$), we have that $h^1(\mathcal{O}_L(D))=0$.

From the cohomology sequence we get

$$0=H^1(D'') \to H^1(D) \to H^1(\mathcal{O}_L(D))=0,$$

and so $h^1(D)=0$ and we are done for $k=1$.

Now let there be $k \geqslant 2$, and consider another way to split the divisor D: $D=D'+D''$, with $D'=(m_k-1)E_0-(m_k-1)E_k$, $D''=(t-m_k+1)E_0-m_1E_1-\cdots-m_{k-1}E_{k-1}-E_{k+1}-\cdots-E_s$.

Let $L \in |D'|$ be the disjoint union of m_k-1 rational curves $L=L_1 \cup \cdots \cup L_{m_k-1}$, where the L_i's are strict transforms of m_k-1 distinct lines through P_k, $L_i \in |E_0-E_k|$, and consider the exact sequence

$$(**) \qquad 0 \to \mathcal{O}_X(D'') \to \mathcal{O}_X(D) \to \mathcal{O}_L(D) \to 0.$$

We have $h^1(D'')=0$ by the induction hypothesis. Moreover $\forall i=1, ..., m_k-1$ we have $\deg \mathcal{O}_{L_i}(D)=D \cdot L_i=t-m_k>0$. So $h^1(\mathcal{O}_{L_i}(D))=0$, and $h^1(\mathcal{O}_{L_i}(D))=\sum_{i=1}^{m_k-1} h^1(\mathcal{O}_{L_i}(D))=0$.

Hence we can conclude, by $(**)$, that $h^1(D)=0$. ∎

We are now ready to prove the main result.

Proof of the Theorem. Suppose $D \neq mC$, $\forall m \geqslant 2$, and let $D=(d+n)E_0-\sum_{i=1}^{s} m_iE_i$ and $j=\max\{i \mid m_i>0\}$; we can assume $j=s$, because otherwise we can just work on the surface $X_j = \{\text{blow-up of } \mathbb{P}^2 \text{ at } P_1, ..., P_j\}$, by (0.3).

By Lemma 1.3, we have an expression

$$D = b_0 E_0 + b_1 S_1 + b_2 S_2 + \cdots + b_s S_s,$$

where $b_i \geqslant 0$ $\forall i = 0, ..., s-1$, and $b_s > 0$ by our supposition ($j = s$).

We shall work by induction on $b(D) = b_{3d-1} + \cdots + b_s$. If $b(D) = 0$ then $s < 3d - 1$ and we are done by Lemma 1.5.

Now let $b(D) > 0$. Consider the exact sequence (note that $C \in |S_s|$)

$$0 \to \mathcal{O}_X(D - S_s) \to \mathcal{O}_X(D) \to \mathcal{O}_C(D) \to 0.$$

The divisor $D - S_s$ is d-standard by Lemma 1.3 (it is a combination of the S_is with non-negative coefficients) and is $\not\equiv mC$, $m \geqslant 2$, since D is not of that form. Hence, by the induction hypothesis, $h^1(D - S_s) = 0$. Moreover, by (0.2), we have $h^2(D - S_s) = 0$. Passing to the cohomology sequence, we get

$$0 = H^1(D - S_s) \to H^1(D) \to H^1(\mathcal{O}_C(D)) \to H^2(D - S_s) = 0,$$

and we shall be done if we can prove that $H^1(\mathcal{O}_C(D)) = 0$.

The arithmetic genus of C is given by the adjunction formula $2p_\alpha - 2 = C(C + K_X) = d^2 - 3d$. So to conclude that $h^1(\mathcal{O}_C(D)) = 0$ it will be sufficient, by (0.3), to check that

$$\deg \mathcal{O}_C(D) = D \cdot C > 2p_\alpha - 2 = d^2 - 3d.$$

If $d = 3$, $h^1(C) = 0$ implies that $s \leqslant 9$, and $D \cdot C = -D \cdot K_X > 0 = d^2 - 3d$, unless $D = -mK_X$, but this is the case we have excluded, since $-K_x = C$.

Hence we are done if $d = 3$. Now if $d \geqslant 4$, what we have to prove is

$$C \cdot D = (d+n)\,d - \sum_{i=1}^{s} m_i \geqslant d^2 - 3d - 1, \tag{1}$$

i.e.,

$$n \geqslant \frac{1}{d}\left(\sum_{i=1}^{s} m_i - 3d + 1\right). \tag{2}$$

We always have $\sum_{i=1}^{s} m_i \leqslant s/d \sum_{i=1}^{d} m_i$, and we have an equality only if $m_1 = \cdots = m_s$.

Moreover, the hypothesis $h^1(C) = 0$ implies that $s \leqslant d(d+3)/2$. So we get $\sum_{i=1}^{s} m_i \leqslant (\sum_{i=1}^{d} m_i)(d+3)/2$.

Let $A = \sum_{i=1}^{d} m_i$. In order to have (2) it is enough that

$$n \geqslant \frac{1}{2d}(A(d+3) - 6d + 2).$$

But we know that $n \geqslant A - d$ (D is d-standard), so we are done if $2d(A-d) \geqslant [A(d+3) - 6d + 2]$, i.e., if

$$A \geqslant 2d + \frac{2}{d-3}. \tag{3}$$

From (3) we see immediately that we are done if $d \geqslant 5$ and $A \geqslant 2d+1$.

If $d=4$, (3) is always true for $A \geqslant 2d+2=10$.

If $A=9$, consider $\sum_{i=1}^{s} m_i$; it is always $\leqslant 29$ (it is 29 when $m_1=3$, $m_2 = \cdots = m_s = 2$).

So (2) gives $n \geqslant (29-12+1)/4 = 9/2$. But $n \geqslant 5$, since D is 4-standard, so (3) is satisfied and we are also done in this case.

Hence we are done when $A \geqslant 2d+1$. If $A < 2d$, then D can be written as

$$D = (d+n)\, E_0 - m_1 E_1 - \cdots - m_k E_k - E_{k+1} - \cdots E_s,$$

where $k < d$ and $d+n \geqslant m_1 - \cdots - m_s + d - k$. So we are in the hypothesis of Lemma 1.6, and $h^i(D) = 0$, as we want.

The only problem left is for $A = 2d$. In this case we have

$$D \cdot C = (d+n)\, d - \sum_{i=1}^{s} m_i \geqslant 2d^2 - \frac{A \cdot s}{d}$$

$$\geqslant 2d^2 - \frac{2d}{d} \cdot \frac{d(d+3)}{2} = d^2 - 3d.$$

But $D \cdot C = d^2 - 3d$ only if $D = 2dE_0 - 2E_1 - \cdots - 2E_s = 2C$ and $s = d(d+3)/2$, i.e., in a case that we have excluded from the beginning. So we must have $D \cdot C \geqslant d^2 - 3d + 1 = 2p_\alpha - 1$ and $h^1(\mathcal{O}_C(D)) = 0$. ∎

Remark 1.7. Note that in the proof of Theorem (2.1) we have used the hypothesis $h^1(dE_0 - E_1 - \cdots - E_s) = 0$ only if $A = \sum_{i=1}^{d} m_i \leqslant 2d$.

Hence, if we assume that D is such that $A > 2d$, we can drop that hypothesis, and just ask instead that $s \leqslant d(d+3)/2$.

Criteria of irreducibility and very ampleness for d-standard divisors, and applications and examples can be found in [Gi].

Let us now consider the particular case left out by Theorem 1.1, i.e., let us prove Proposition 1.2:

Proof of the Proposition. First we prove that $d > 3$ implies that $n = 2$. Hence, if $d \geqslant 4$ and $m > 2$, we want to show that $h^1(mC) \neq 0$ implies $h^1((m-1)\,C) \neq 0$ also.

Consider the exact sequence

$$(**) \qquad 0 \to \mathcal{O}_X((m-1)\,C) \to \mathcal{O}_X(mC) \to \mathcal{O}_C(mC) \to 0.$$

Since $\deg \mathcal{O}_C(mC) = md^2 - m \cdot d(d+3)/2 = m \cdot (d^2 - 3d)/2 > d^2 - 3d$ and $2p_\alpha - 2 = d^2 - 3d$, we have $h^1(\mathcal{O}_C(mC) = 0$ by 0.3, so if $h^1(mC) \neq 0$ it must be $h^1((m-1)\,C) \neq 0$, by $(**)$.

We can proceed in this way until we get to $h^1(2C) \neq 0$, hence $n = 2$ ($n \neq 1$ by hypothesis).

Note that $\deg \mathcal{O}_C(2C) = 2p_\alpha - 2$ and $h^1(2D) \neq 0$ imply (by (0.3)) that $\mathcal{O}_C(2C) = \omega_c$, as required.

In the case $d = 3$ we have $p_\alpha(C) = 1$ and $\deg \mathcal{O}_C(mC) = 0$ for any m. From $h^1((n-1)\,C) = h^2((n-1)\,C) = 0$, we have (using $(**)$ for $m = n$) that $h^1(\mathcal{O}_C(nC) \neq 0$, so we can apply (0.3) again to get $\mathcal{O}_C(nC) = \omega_C$. ∎

2. *Linear Systems of Plane Curves*

In this section we shall prove the results (Theorem 1 and Proposition 2) announced in the introduction.

Theorem 1 is a direct consequence of Theorem 1.1; in fact, the strict transform $C' \subseteq X$ of $C \subseteq \mathbb{P}^2$ will satisfy the requirements of Theorem 1.1, and to say that

$$h^1(tE_0 - m_1 E_1 - \cdots - m_s E_s) = 0 \qquad \text{for} \quad t \geqslant \sum_{i=1}^{d} m_i$$

amounts to saying, as we have seen, that $\tau(Z) \leqslant \sum_{i=1}^{d} m_i$.

Now let us consider the special case in Theorem 1. We have the following:

PROPOSITION 2.1. *Let $P_1, ..., P_s$, C, Z, be as in Theorem* 1, *and let $s = d(d+3)/2$, $m_1 = m_2 = \cdots = m_s = m \geqslant 2$ with $\tau(Z) = dm + 1$. Then*:

(1) *If $d = 3$ there exists a curve $C' \subseteq \mathbb{P}^2$, of degree $3n$ with $1 \leqslant n \leqslant m$, cutting on C the divisor $nP_1 + nP_2 + \cdots + nP_9$.*

(2) *If $d \geqslant 4$ there is a curve $C' \subseteq \mathbb{P}^2$, of degree $d+3$, such that C' cuts the divisor $\sum_{i=1}^{s} 2P_i$ on C.*

Proof. Let us consider the blowing-up X of $\mathbb{P}^2$ at the P_is, the divisor $D' = dmE_0 - mE_1 - \cdots - mE_s \in \operatorname{Div} X$, and the strict transform $\tilde{C} \in |dE_0 - E_1 - \cdots - E_s|$ of C on X.

The hypotheses of the proposition imply that $h^1(D') \neq 0$, and so, if

$D = nC$ and n is defined as in the previous section, by Proposition 1.2 we have $h^1(D) \neq 0$ and $\mathcal{O}_{\tilde{C}}(D) = \omega_{\tilde{C}}$.

Since C is a plane curve, $\omega_C = \mathcal{O}_C(d-3)$. So we have also that $\mathcal{O}_{\tilde{C}}((d-3)\,E_0) = \omega_{\tilde{C}}$ on $\tilde{C}$. Therefore the divisor $D-(d-3)\,E_0 = [(n-1)\,d+3]\,E_0 - \sum_{i=1}^{s} mE_i$ cuts on $\tilde{C}$ the zero divisor; i.e., (since $\tilde{C}$ is isomorphic to C) the divisor $\sum_{i=1}^{s} mP_i$ is linearly equivalent on C to $(n-1)\,d+3$ times the hyperplane section, so it is cut by some curve C' of degree $(n-1)\,d+3$, as required. ▮

Now we want to show that the situation described in the previous proposition is a particular one; i.e., that it is not what happens for a *generic* set of points.

We will say that a property is satisfied by a generic set $\{P_1, ..., P_s\}$ of (distinct) points in $\mathbb{P}^2$ if there is a non-empty open subset in the Hilbert scheme Hilb^2_S which parametrizes sets of points with that property.

Let Z be as in Prop. (2.1); then:

PROPOSITION 2.2. *We always have $\tau(Z) \leqslant dm$ for a generic set of points $P_1, ..., P_s$, $s = d(d+3)/2$.*

Proof. Let the notation be as in the proof of Prop. 2.1.

If $d=3$ the condition $\tau(Z) = 3m+1$ is equivalent to asking that the divisor $mP_1 + \cdots + mP_9$ be canonical, i.e., that the order of the element $P_1 + \cdots + P_9$ in the group law of the cubic C be less than or equal to m. This is a closed condition. So for a generic set of nine points on C we have $\tau(Z) = 3m$.

Note that if the base field is $\bar{F}_p$, then every set of points $P_1, ..., P_s$ on C has finite order (see e.g. [Ma]). So for any set of nine points on C there must exist $m > 0$ for which we have $\tau(I) = 3m+1$.

If, instead, the base field has characteristic 0, the elements of finite order in the group of C form a subgroup Λ isomorphic to $\mathbb{Q}^2/\mathbb{Z}^2$ (see e.g. [Ro]). If $P_1 + \cdots + P_9 \notin \Lambda$, then $mP_1 + \cdots + mP_9 \neq K_C$ for every $m > 0$ (this happens for "most" of the points if the field is uncountable, but not for a generic choice of them; i.e., the set of these "very good" 9-tuples of points is not open).

If $d \geqslant 4$ the conditions imply that $2P_1 + \cdots + 2P_s$ is the canonical divisor on C.

Consider the map $\varphi\colon C_{(2g-2)} \to \mathrm{Pic}^{2g-2}C$, where $C_{(n)}$ is the nth symmetric product of C.

The image, under φ, of a group of points that constitute special divisors is just a point in $\mathrm{Pic}^{2g-2}C$, corresponding to the canonical class (see e.g. [Gu]). Therefore the generic divisor of order $2g-2$ on C is not K_C (note that φ is surjective since $2g-2 > g$ because the geometric genus is

$g = p_\alpha > 1$). This implies that for a generic set of points we get $\tau(Z) = dm$. ∎

Note that this proposition and Theorem 1 imply Proposition 2 given in the introduction; in fact, s generic points with $s < \binom{d+2}{2}$ can be assumed to impose independent conditions to curves of degree d and to be smooth points of one of them which is reduced and irreducible.

Let Z, $m_1, ..., m_s$, A_d be as in the Introduction; then for $d = \tau(S) + 1$ a curve C as required in Theorem 1 surely exists (see e.g. [DG.2]), so we have:

COROLLARY 2.3. $\tau(Z) \leqslant A_{\tau(S)+1}$.

Note also that, by Remark 1.8, we have:

PROPOSITION 2.4. *Let $P_i, ..., P_s$ be simple points of an irreducible curve $C \subseteq \mathbb{P}^2$ of degree $d \geqslant 3$. Then if $s \leqslant d(d+3)/2$ and Z is such that $A_d > 2d$, then $\tau(Z) \leqslant A_d$.*

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